# SMD Test Problems for Single-Objective Bilevel Optimization

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## 1 Introduction

This file provides a description to the 12 SMD [1] test problems that are single-objective scalable bilevel optimization problems. The SMD test-suite includes eight unconstrained and four constrained problems. All the problems are scalable in terms of the number of variables at upper and lower levels. The codes for SMD test problems may be accessed from the website `http://bilevel.org`.

To begin with, we provide an introduction to general bilevel optimization problems. Thereafter, we introduce the properties and structure of the SMD test-suite. This is followed by a description of individual test problems. Finally, we summarise the SMD test problems in a table and discuss some necessary precautions while using the test-suite.

## 2 Bilevel Optimization Problems

A bilevel optimization problem involves two levels of optimization tasks, where one level is nested within the other. The outer optimization task is usually called upper level optimization task, and the inner optimization task is called lower level optimization task. The hierarchical structure of the problem requires that only the optimal solutions of the inner optimization task are acceptable as feasible members for the outer optimization task. The problem contains two types of variables; namely the upper level variables $\mathbf{x}_u$, and the lower level variables $\mathbf{x}_l$. The lower level is optimized with respect to the lower level variables $\mathbf{x}_l$, and the upper level variables $\mathbf{x}_u$ act as parameters. An optimal lower level vector and the corresponding upper level vector $\mathbf{x}_u$ constitute a feasible upper level solution, provided the upper level constraints are also satisfied. The upper level problem involves all variables $\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_l)$, and the optimization is to be performed with respect to both $\mathbf{x}_u$ and $\mathbf{x}_l$. In the following, we provide two equivalent formulations for a general bilevel optimization problem with one objective at both levels:

**Definition 1 (Bilevel Optimization Problem (BLOP))** *Let $X = X_U \times X_L$ denote the product of the upper-level decision space $X_U$ and the lower-level decision space $X_L$, i.e. $\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_l) \in X$, if $\mathbf{x}_u \in X_U$ and $\mathbf{x}_l \in X_L$. For upper-level objective function $F : X \to \mathbb{R}$ and lower-level objective function $f : X \to \mathbb{R}$, a general bilevel optimization problem is given by*

$$\begin{aligned} \underset{\mathbf{x} \in X}{\mathit{Min}} \quad & F(\mathbf{x}), \\ \mathit{s.t.} \quad & \mathbf{x}_l \in \underset{\mathbf{x}_l \in X_L}{\operatorname{argmin}} \left\{ f(\mathbf{x}) \mid g_i(\mathbf{x}) \geq 0, i \in I \right\}, \\ & G_j(\mathbf{x}) \geq 0, j \in J. \end{aligned} \tag{1}$$

*where the functions $g_i : X \to \mathbb{R}$, $i \in I$, represent lower-level constraints and $G_j : X \to \mathbb{R}$, $j \in J$, is the collection of upper-level constraints.*

In the above formulation, a vector $\mathbf{x}^{(0)} = (\mathbf{x}_u^{(0)}, \mathbf{x}_l^{(0)})$ is considered feasible at the upper level, if it satisfies all the upper level constraints, and vector $\mathbf{x}_l^{(0)}$ is optimal at the lower level for the given $\mathbf{x}_u^{(0)}$. We observe in this formulation that the lower-level problem is a parameterized constraint to the upper-level problem. An equivalent formulation of the bilevel optimization problem is obtained by replacing the lower-level optimization problem with a set value function which maps the given upper-level decision vector to the corresponding set of optimal lower-level solutions. In the domain of Stackelberg games, such mapping is referred as the rational reaction of the follower to the leader's choice $\mathbf{x}_u$.

**Definition 2 (Alternative definition of Bilevel Problem)** *Let set-valued function $\Psi : X_U \rightrightarrows X_L$, denote the optimal-solution set mapping of the lower level problem, i.e.*

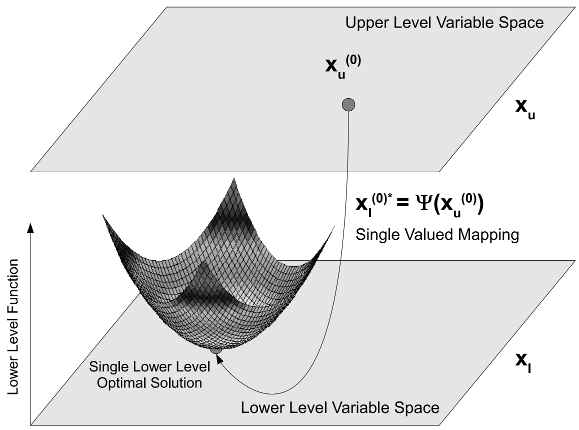
$$\Psi(\mathbf{x}_u) = \underset{\mathbf{x}_l \in X_L}{\operatorname{argmin}} \left\{ f(\mathbf{x}) \mid g_i(\mathbf{x}) \geq 0, i \in I \right\}.$$

*A general bilevel optimization problem (BLOP) is then given by*

$$\begin{aligned} \underset{\mathbf{x} \in X}{\mathit{Min}} \quad & F(\mathbf{x}), \\ \mathit{s.t.} \quad & \mathbf{x}_l \in \Psi(\mathbf{x}_u), \\ & G_j(\mathbf{x}) \geq 0, j \in J. \end{aligned} \tag{2}$$

*where the function $\Psi$ may be a single-vector valued or a multi-vector valued function depending on whether the lower level function has multiple global optimal solutions or not.*

In the test problem construction procedure, the $\Psi$ function provides a convenient description of the relationship between the upper and lower level problems. Figures 1 and 2 illustrate two scenarios, where $\Psi$ can be a single vector valued or a multi-vector valued function respectively. In Figure 1, the lower level problem is shown to be a paraboloid with a single minimum function value corresponding to the set of upper level variables $\mathbf{x}_u$. Figure 2 represents a scenario where the lower level function is a paraboloid sliced from the bottom with a horizontal plane. This leads to multiple minimum values for the lower level problem, and therefore, multiple lower level solutions correspond to the set of upper level variables $\mathbf{x}_u$.



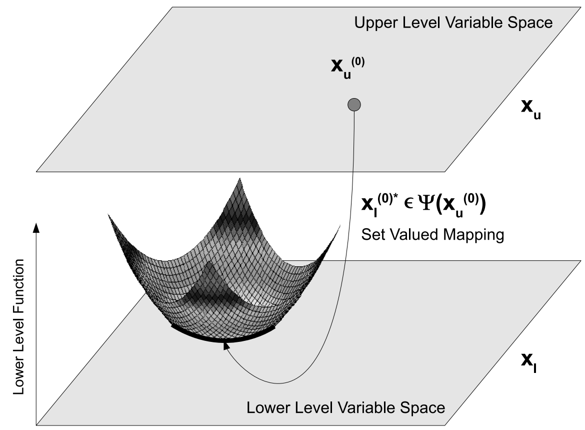




Figure 1: Relationship between upper and lower level variables in case of a single-vector valued mapping. For simplicity the lower level function has the shape of a paraboloid.

Figure 2: Relationship between upper and lower level variables in case of a multi-vector valued mapping. The lower level function is shown in the shape of a paraboloid with the bottom sliced with a plane.

# 3 Properties of SMD test problems

The SMD test problems provide a mix of various kinds of difficultes that can be encountered in bilevel optimization. All the SMD test problems are single-objective minimization problems at both levels. All the SMD test problems represent one or more of the following properties.

1. The optimal solution of the problems are known.
2. Relationship between the lower level optimal solutions and the upper level variables is clearly identified.
3. Controllable difficulty in convergence at upper and lower levels.
4. Controllable difficulty caused by interaction of the two levels.
5. Multiple global solutions at the lower level for a given set of upper level variables.
6. Conflict or cooperation between the two levels.
7. Scalability to any number of decision variables at upper and lower levels.
8. Constraints (scalable and non-scalable) at upper and lower levels.

# 4 Structure of SMD Problems

In order to have a tractable structure, the SMD test problems have the upper and lower level functions splitted into three components. The upper and lower level variables have also been splitted into two components. Each of the components is specialized for induction of certain kinds of difficulties into the bilevel problem. A summary on the roles of different components is provided in Table 1.

Table 1: Overview of test-problem framework components

Panel A: Decomposition of decision variables

| Upper-level variables | | Lower-level variables | |
|---|---|---|---|
| Vector | Purpose | Vector | Purpose |
| $\mathbf{x}_{u1}$ | Complexity on upper-level | $\mathbf{x}_{l1}$ | Complexity on lower-level |
| $\mathbf{x}_{u2}$ | Interaction with lower-level | $\mathbf{x}_{l2}$ | Interaction with upper-level |

Panel B: Decomposition of objective functions

| Upper-level objective function | | Lower-level objective function | |
|---|---|---|---|
| Component | Purpose | Component | Purpose |
| $F_1(\mathbf{x}_{u1})$ | Difficulty in convergence | $f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2})$ | Functional dependence |
| $F_2(\mathbf{x}_{l1})$ | Conflict / co-operation | $f_2(\mathbf{x}_{l1})$ | Difficulty in convergence |
| $F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$ | Difficulty in interaction | $f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$ | Difficulty in interaction |

$$\begin{array}{l} F(\mathbf{x}_u, \mathbf{x}_l) = F_1(\mathbf{x}_{u1}) + F_2(\mathbf{x}_{l1}) + F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2}) \\ f(\mathbf{x}_u, \mathbf{x}_l) = f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2}) + f_2(\mathbf{x}_{l1}) + f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2}) \\ \text{where} \\ \qquad \mathbf{x}_u = (\mathbf{x}_{u1}, \mathbf{x}_{u2}) \quad \text{and} \quad \mathbf{x}_l = (\mathbf{x}_{l1}, \mathbf{x}_{l2}) \end{array} \tag{3}$$

# 5 SMD test problems

In this section, we provide a description for each of the SMD test problems. Each problem represents a different difficulty standard in terms of convergence at the two levels, complexity of interaction between the two levels, and multi-modalities at each of the levels. The first eight problems are unconstrained and the remaining four are constrained. All the problems are minimization problems at both levels.

## 5.1 SMD1

This is a simple test problem, where the lower level problem is a convex optimization task and the upper level is convex with respect to upper level variables and optimal lower level variables. The two levels cooperate with each other. The constituent functions are chosen as

$$\begin{array}{l} F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\ F_2 = \sum_{i=1}^{q} (x_{l1}^i)^2, \\ F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 + \sum_{i=1}^{r} (x_{u2}^i - \tan x_{l2}^i)^2, \\ f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\ f_2 = \sum_{i=1}^{q} (x_{l1}^i)^2, \\ f_3 = \sum_{i=1}^{r} (x_{u2}^i - \tan x_{l2}^i)^2. \end{array} \tag{4}$$

The range of variables is as follows:

$$\begin{array}{l} x_{u1}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, p\}, \\ x_{u2}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, r\}, \\ x_{l1}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, q\}, \\ x_{l2}^i \in (\frac{-\pi}{2}, \frac{\pi}{2}), \;\; \forall \; i \in \{1, 2, \ldots, r\}. \end{array} \tag{5}$$

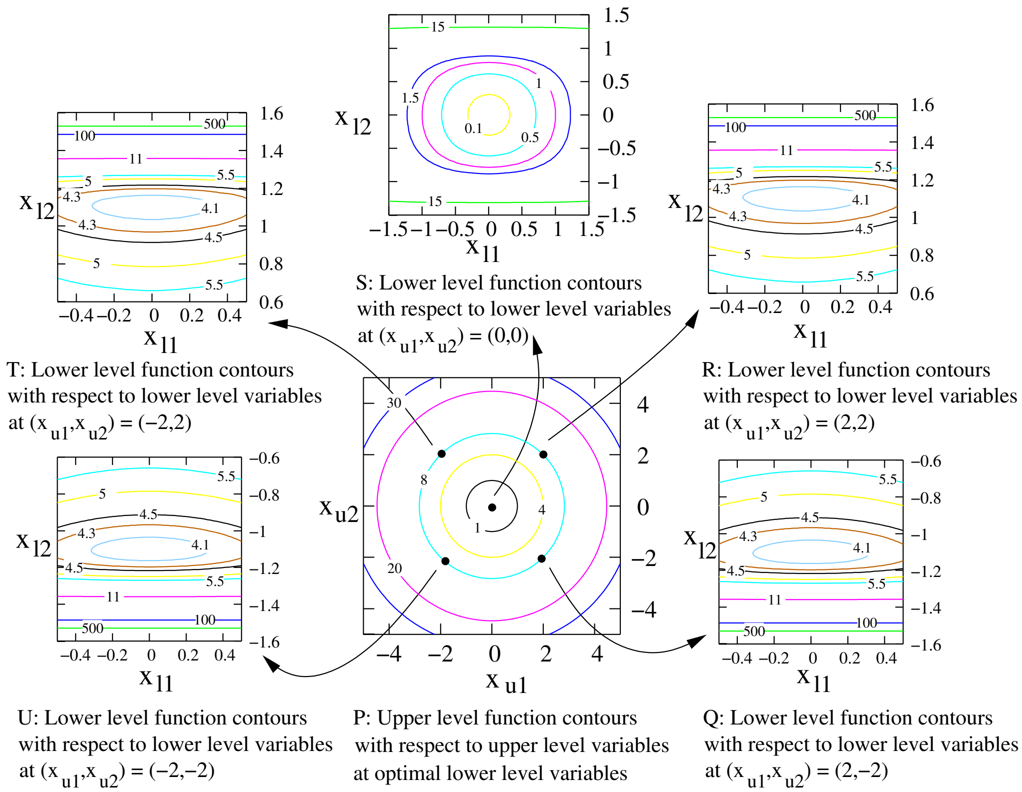



Figure 3: Upper and lower level function contours for a four-variable SMD1 test problem.

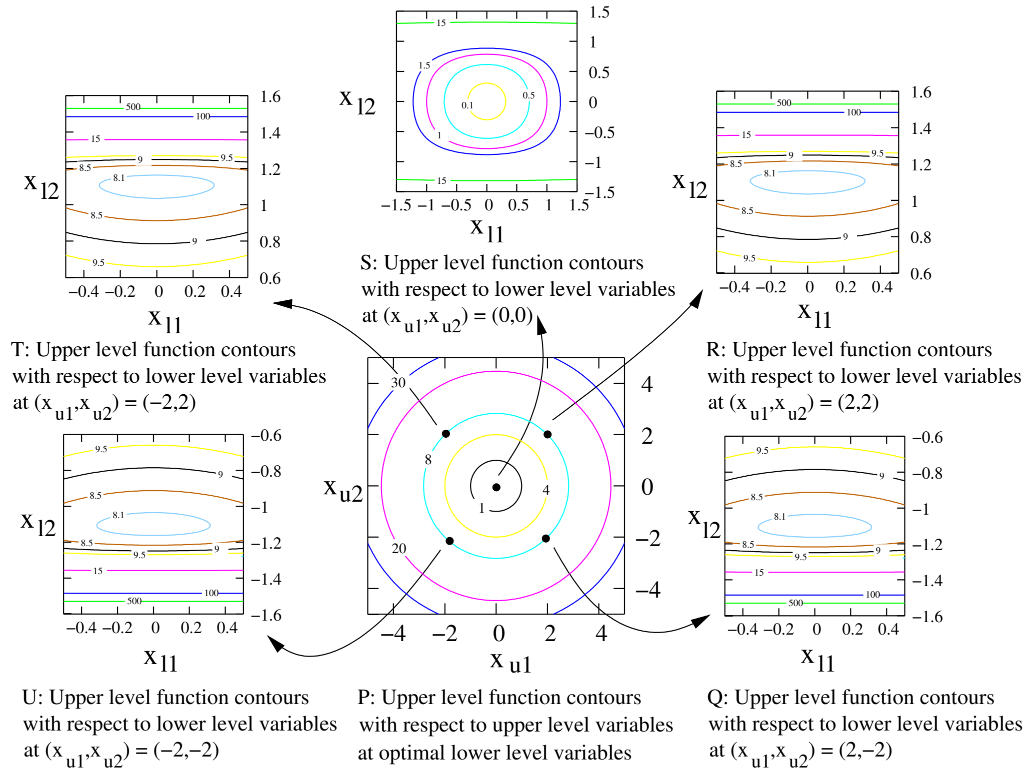



Figure 4: Upper level function contours for a four-variable SMD1 test problem.

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{aligned}
x_{l1}^i &= 0, \;\; \forall \;\; i \in \{1, 2, \ldots, p\}, \\
x_{l2}^i &= \tan^{-1} x_{u2}^i, \;\; \forall \;\; i \in \{1, 2, \ldots, r\}.
\end{aligned} \tag{6}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

Figure 3 shows the contours of the upper and lower level functions with respect to the upper and lower level variables for a four-variable test problem. The problem has two upper level variables and two lower level variables, such that the dimensions of $\mathbf{x}_{u1}, \mathbf{x}_{u2}, \mathbf{x}_{l1}$ and $\mathbf{x}_{u2}$ are all one. Sub-figure P shows the upper level function contours with respect to the upper level variables, assuming that the lower level variables are at the optima. Fixing the upper level variables $(\mathbf{x}_{u1}, \mathbf{x}_{u2})$ at five different locations, i.e. $(2, 2), (-2, 2), (2, -2), (-2, -2)$ and $(0, 0)$, the lower level function contours are shown with respect to the lower level variables. This shows that the contours of the lower level optimization problem may be different for different upper level vectors.

Figure 4 shows the contours of the upper level function with respect to the upper and lower level variables. Sub-figure P once again shows the upper level function contours with respect to the upper level variables. However, sub-figures Q, R, S, T and V now represent the upper level function contours at different $(\mathbf{x}_{u1}, \mathbf{x}_{u2})$, i.e. $(2, 2), (-2, 2), (2, -2), (-2, -2)$ and $(0, 0)$. From sub-figures Q, R, S, T and V, we observe that if the lower level variables move away from its optimal location, the upper level function value deteriorates. This means that the upper level function and the lower level functions are cooperative.

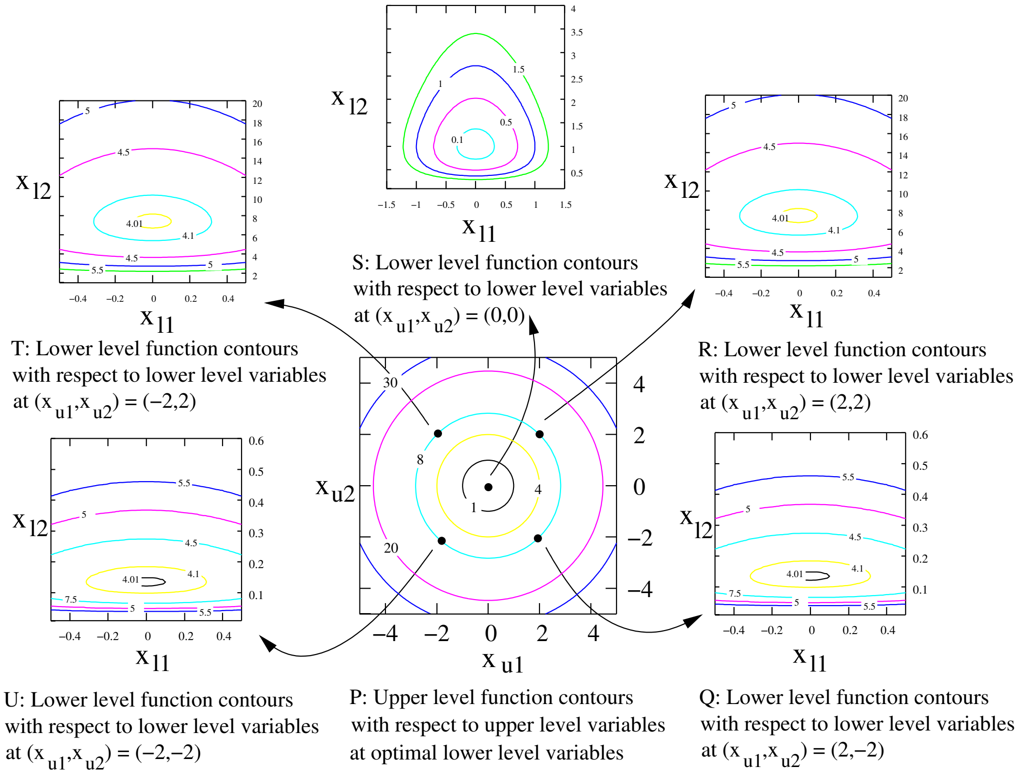



Figure 5: Upper and lower level function contours for a four-variable SMD2 test problem.

## 5.2 SMD2

This test problem is similar to the SMD1 test problem. However, there is a conflict between the upper level and lower level optimization task. The lower level optimization problem is once again a convex optimization task and the upper level optimization is convex with respect to upper level variables and optimal lower level variables. Since the two levels are conflicting, an inaccurate lower level optimum may lead to upper level function value better than the true optimum for the bilevel problem. The constituent functions are chosen as

$$\begin{aligned}
F_1 &= \textstyle\sum_{i=1}^{p} (x_{u1}^i)^2, \\
F_2 &= -\textstyle\sum_{i=1}^{q} (x_{l1}^i)^2, \\
F_3 &= \textstyle\sum_{i=1}^{r} (x_{u2}^i)^2 - \sum_{i=1}^{r} (x_{u2}^i - \log x_{l2}^i)^2, \\
f_1 &= \textstyle\sum_{i=1}^{p} (x_{u1}^i)^2, \\
f_2 &= \textstyle\sum_{i=1}^{q} (x_{l1}^i)^2, \\
f_3 &= \textstyle\sum_{i=1}^{r} (x_{u2}^i - \log x_{l2}^i)^2.
\end{aligned} \tag{7}$$

The range of variables is as follows:

$$\begin{aligned}
x_{u1}^i &\in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, p\}, \\
x_{u2}^i &\in [-5, 1], \ \ \forall \ i \in \{1, 2, \ldots, r\}, \\
x_{l1}^i &\in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
x_{l2}^i &\in (0, e], \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{aligned} \tag{8}$$

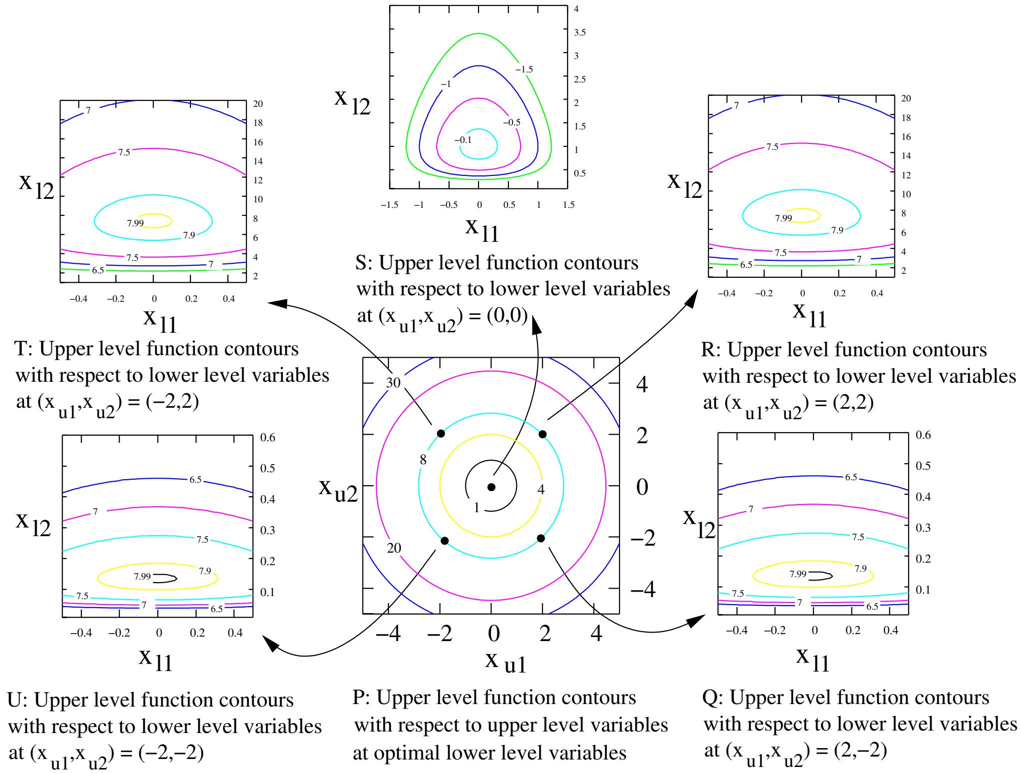



Figure 6: Upper level function contours for a four-variable SMD2 test problem.

Relationship between upper level variables and lower level optimal variables is given as follows:

$$\begin{aligned} x_{l1}^i &= 0, \;\; \forall \;\, i \in \{1, 2, \ldots, q\}, \\ x_{l2}^i &= \log^{-1} x_{u2}^i, \;\; \forall \;\, i \in \{1, 2, \ldots, r\}. \end{aligned} \tag{9}$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

Figure 5 shows the contours of the upper and lower level functions with respect to the upper and lower level variables for a four-variable test problem. The problem has two upper level variables and two lower level variables, such that the dimension of $\mathbf{x}_{u1}, \mathbf{x}_{u2}, \mathbf{x}_{l1}$ and $\mathbf{x}_{u2}$ are all one. The figure provides the same information about SMD2, as Figure 3 provides about SMD1. However, the shape of the contours differ, which is caused by the use of different $F_3$ and $f_3$ functions.

Figure 6 shows the contours of the upper level function with respect to the upper and lower level variables, and provides the same information as Figure 4 provides about SMD1. This figure shows the conflicting nature of the problem caused by using a negative sign in $F_2$. The conflicting nature can be observed from the sub-figures Q, R, S, T and U. For a given $\mathbf{x}_u$, as one moves away from the lower level optimal solution, the upper level function value further reduces. On the other hand, in Figure 5 we observe that moving away from the lower level optimal solution causes an increase in lower level function value.

## 5.3 SMD3

In this test problem there is a cooperation between the two levels. The difficulty is introduced in terms of multi-modality at the lower level which contains the Rastrigin's function. The upper level is convex with respect to upper level variables and optimal lower level variables. The constituent functions are chosen as

$$
\begin{aligned}
&F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\
&F_2 = \sum_{i=1}^{q} (x_{l1}^i)^2, \\
&F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 + \sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2, \\
&f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\
&f_2 = q + \sum_{i=1}^{q} \left( (x_{l1}^i)^2 - \cos 2\pi x_{l1}^i \right), \\
&f_3 = \sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2.
\end{aligned}
\tag{10}
$$

The range of variables is as follows:

$$
\begin{aligned}
&x_{u1}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, p\}, \\
&x_{u2}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, r\}, \\
&x_{l1}^i \in [-5, 10], \;\; \forall \; i \in \{1, 2, \ldots, q\}, \\
&x_{l2}^i \in (\tfrac{-\pi}{2}, \tfrac{\pi}{2}), \;\; \forall \; i \in \{1, 2, \ldots, r\}.
\end{aligned}
\tag{11}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{aligned}
&x_{l1}^i = 0, \;\; \forall \; i \in \{1, 2, \ldots, q\}, \\
&x_{l2}^i = \tan^{-1}(x_{u2}^i)^2, \;\; \forall \; i \in \{1, 2, \ldots, r\}.
\end{aligned}
\tag{12}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima. Rastrigin's function used in $f_2$ has multiple local optima around the global optimum, which introduces convergence difficulties at the lower level.

Sub-figure P in Figure 7 shows the contours of the upper level function with respect to the upper level variables assuming the lower level variables to be optimal at each $\mathbf{x}_u$. Sub-figures Q, R, S, T, and U show the behavior of the lower level function at 5 different locations of $\mathbf{x}_u$, which are $(2, 2), (-2, 2), (2, -2), (-2, -2)$ and $(0, 0)$. The problem is once again assumed to have two upper level variables and two lower level variables, such that the dimensions of $\mathbf{x}_{u1}, \mathbf{x}_{u2}, \mathbf{x}_{l1}$ and $\mathbf{x}_{u2}$ are all one. The figure shows that there is a different lower level optimization problem at each $\mathbf{x}_u$ which is required to be solved in order to achieve a feasible solution at the upper level. The contours of the lower level optimization problem differ based on the location of upper level vector. It can be observed that the Rastrigin's function at the lower level introduces multiple local optima into the problem. The contours of the lower level are further distorted because of the presence of the $\tan(\cdot)$ function at the lower level.

In spite of multiple local optima at the lower level, this problem is easier to solve because of the cooperating nature of the functions at the two levels. If a lower level optimization problem is stuck at a local optimum for a particular $\mathbf{x}_u$ (say $\mathbf{x}_u^{(0)}$), it will have a poorer objective function value at the upper level. However, as soon as another lower level optimization problem is solved in the vicinity of $\mathbf{x}_u^{(0)}$, which attains a global lower level optimum, then it will have a better objective function value at the upper level and will dominate the previous inaccurate solution. Therefore, a method which is able to handle multi-modality at the lower level at least in few of its lower level optimization runs will be able to successfully solve this problem.

## 5.4 SMD4

In this test problem there is a conflict between the two levels. The difficulty is in terms of multi-modality at the lower level which once again contains the Rastrigin's function. The upper level is

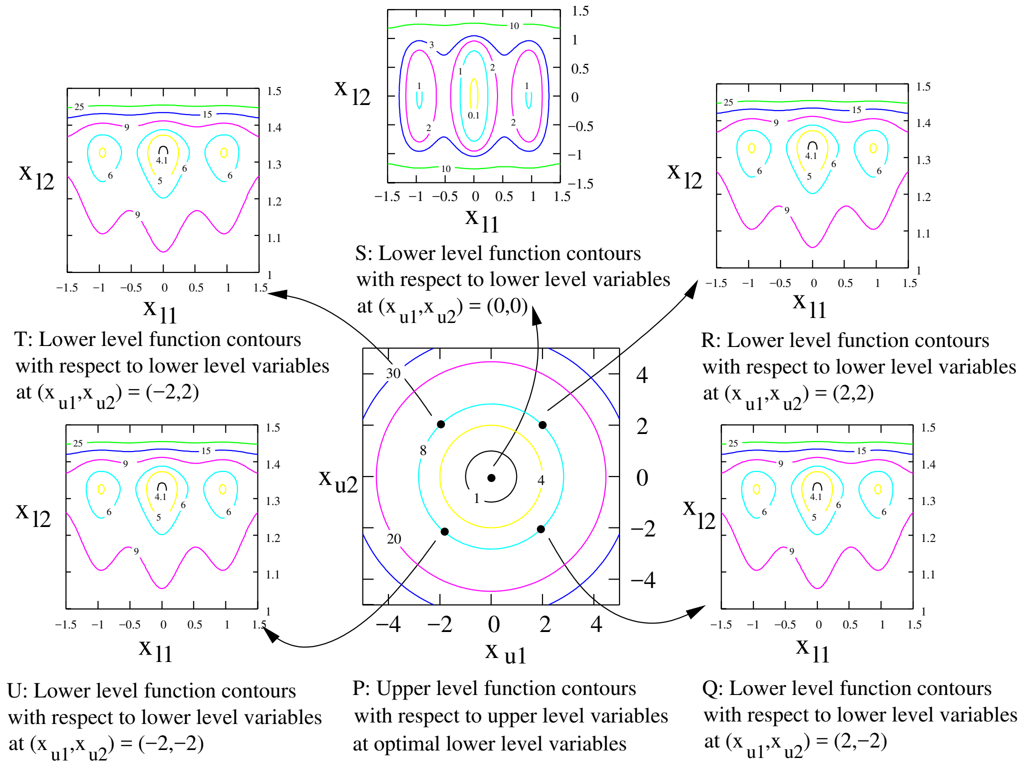


Figure 7: Upper and lower level function contours for a four-variable SMD3 test problem.

convex with respect to upper level variables and optimal lower level variables. The constituent functions are chosen as

$$
\begin{aligned}
F_1 &= \sum_{i=1}^{p}(x_{u1}^i)^2, \\
F_2 &= -\sum_{i=1}^{q}(x_{l1}^i)^2, \\
F_3 &= \sum_{i=1}^{r}(x_{u2}^i)^2 - \sum_{i=1}^{r}(|x_{u2}^i| - \log(1 + x_{l2}^i))^2, \\
f_1 &= \sum_{i=1}^{p}(x_{u1}^i)^2, \\
f_2 &= q + \sum_{i=1}^{q}\left((x_{l1}^i)^2 - \cos 2\pi x_{l1}^i\right), \\
f_3 &= \sum_{i=1}^{r}(|x_{u2}^i| - \log(1 + x_{l2}^i))^2.
\end{aligned} \tag{13}
$$

The range of variables is as follows:

$$
\begin{aligned}
&x_{u1}^i \in [-5, 10], \ \ \forall\ i \in \{1, 2, \ldots, p\}, \\
&x_{u2}^i \in [-1, 1], \ \ \forall\ i \in \{1, 2, \ldots, r\}, \\
&x_{l1}^i \in [-5, 10], \ \ \forall\ i \in \{1, 2, \ldots, q\}, \\
&x_{l2}^i \in [0, e], \ \ \forall\ i \in \{1, 2, \ldots, r\}.
\end{aligned} \tag{14}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{aligned}
&x_{l1}^i = 0, \ \ \forall\ i \in \{1, 2, \ldots, q\}, \\
&x_{l2}^i = \log^{-1}|x_{u2}^i| - 1, \ \ \forall\ i \in \{1, 2, \ldots, r\}.
\end{aligned} \tag{15}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

Figure 8 represents the same information as in Figure 7 for a four-variable bilevel problem. However, this problem involves conflict between the two levels, which makes it significantly more difficult

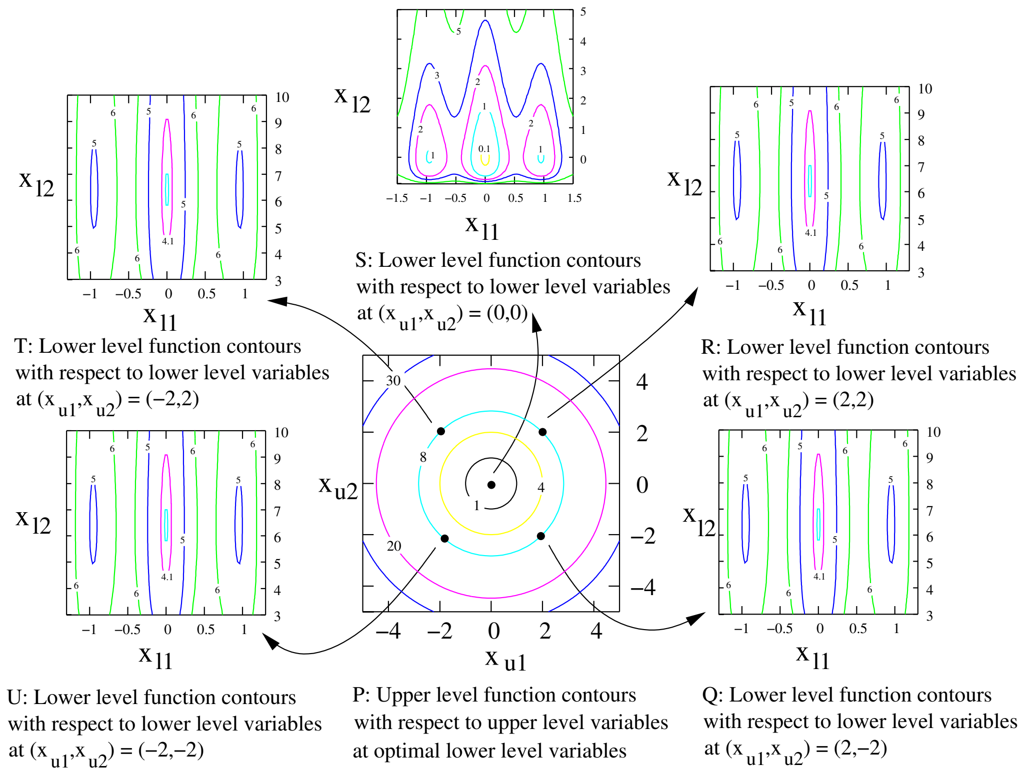



Figure 8: Upper and lower level function contours for a four-variable SMD4 test problem.

to solve than the previous test problem. If a lower level optimization problem is stuck at a local optimum for a particular $\mathbf{x}_u$, it will end up having a better objective function value at the upper level than what it will attain at the true global lower level optimum. Therefore, even if another lower level optimization problem is successfully solved in the vicinity of $\mathbf{x}_u$, the previous inaccurate solution will dominate the new solution at the upper level. This problem can be handled only by those methods which are able to efficiently handle lower level multi-modality without getting stuck in a local basin.

## 5.5 SMD5

In this test problem, there is a conflict between the two levels. The difficulty introduced is in terms of multi-modality and convergence at the lower level. The lower level problem contains the Rosenbrock's (banana) function such that the global optimum lies in a long, narrow, flat parabolic valley. The upper level is convex with respect to upper level variables and optimal lower level variables. The constituent functions are chosen as

$$\begin{aligned}
F_1 &= \textstyle\sum_{i=1}^{p}(x_{u1}^i)^2, \\
F_2 &= -\textstyle\sum_{i=1}^{q-1}\left(\left(x_{l1}^{i+1}-(x_{l1}^i)^2\right)^2+(x_{l1}^i-1)^2\right), \\
F_3 &= \textstyle\sum_{i=1}^{r}(x_{u2}^i)^2-\sum_{i=1}^{r}(|x_{u2}^i|-(x_{l2}^i)^2)^2, \\
f_1 &= \textstyle\sum_{i=1}^{p}(x_{u1}^i)^2, \\
f_2 &= \textstyle\sum_{i=1}^{q-1}\left(\left(x_{l1}^{i+1}-(x_{l1}^i)^2\right)^2+(x_{l1}^i-1)^2\right), \\
f_3 &= \textstyle\sum_{i=1}^{r}(|x_{u2}^i|-(x_{l2}^i)^2)^2.
\end{aligned} \tag{16}$$

The range of variables is as follows:

$$
\begin{array}{l}
x_{u1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, p\}, \\
x_{u2}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, r\}, \\
x_{l1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
x_{l2}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{17}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{array}{l}
x_{l1}^i = 1, \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
x_{l2}^i = \sqrt{|x_{u2}^i|}, \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{18}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

## 5.6 SMD6

In this test problem, there is again a conflict between the two levels. However, this problem differs from the previous problems by containing infinitely many global solutions at the lower level for any given upper level vector. Out of the entire global solution set, there is only a single lower level point which corresponds to the best upper level function value. The constituent functions are chosen as

$$
\begin{array}{l}
F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\
F_2 = -\sum_{i=1}^{q} (x_{l1}^i)^2 + \sum_{i=q+1}^{q+s} (x_{l1}^i)^2, \\
F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 - \sum_{i=1}^{r} (x_{u2}^i - x_{l2}^i)^2, \\
f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2, \\
f_2 = \sum_{i=1}^{q} (x_{l1}^i)^2 + \sum_{i=q+1, i=i+2}^{q+s-1} (x_{l1}^{i+1} - x_{l1}^i)^2, \\
f_3 = \sum_{i=1}^{r} (x_{u2}^i - x_{l2}^i)^2.
\end{array}
\tag{19}
$$

The range of variables is as follows:

$$
\begin{array}{l}
x_{u1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, p\}, \\
x_{u2}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, r\}, \\
x_{l1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, q+s\}, \\
x_{l2}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{20}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{array}{l}
x_{l1}^i = 0, \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
x_{l2}^i = x_{u2}^i, \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{21}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

Figure 9 shows the second term ($(x_{l1}^i - x_{l1}^j)^2$, for $s = 2$) for function $f_2$, and its contours at the lower level. It can be observed from the figure that all the points along $\mathbf{x}_{l1}^j = \mathbf{x}_{l2}^i$ have a value 0 for the function $f_2$. All these points are responsible for introducing multiple global optimal solutions at the lower level for any given upper level variable vector. However, out of all the global optimal solutions at the lower level, the solution $\mathbf{x}_{l1}^j = \mathbf{x}_{l2}^i = 0$ provides the best function value at the upper level for any given upper level variable vector.

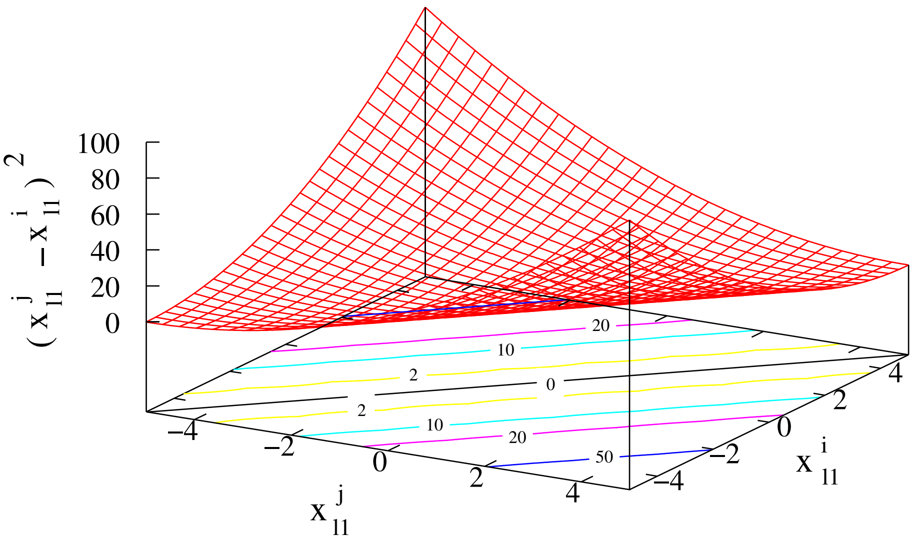



Figure 9: Plot of the term in $f_2$ responsible for creating multiple optimum solutions at the lower level. The value of the term is zero at all the points in the valley.

## 5.7 SMD7

In this test problem, we introduce complexities at the upper level while keeping the lower level optimization task relatively simpler. Most of the previous test problems would be useful for testing the ability of algorithms to handle lower level optimization task efficiently. However, this test problem contains multi-modality at the upper level, which demands a global optimization approach at the upper level. The function $F_1$ at the upper level represents a slightly modified Griewank function. The constituent functions are chosen as

$$\begin{aligned}
&F_1 = 1 + \frac{1}{400}\sum_{i=1}^{p}\left(x_{u1}^i\right)^2 - \Pi_{i=1}^{p}\left(\cos\frac{x_{u1}^i}{\sqrt{i}}\right),\\
&F_2 = -\sum_{i=1}^{q}(x_{l1}^i)^2,\\
&F_3 = \sum_{i=1}^{r}(x_{u2}^i)^2 - \sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2,\\
&f_1 = \sum_{i=1}^{p}(x_{u1}^i)^3,\\
&f_2 = \sum_{i=1}^{q}(x_{l1}^i)^2,\\
&f_3 = \sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2.
\end{aligned} \tag{22}$$

The range of variables is as follows:

$$\begin{aligned}
&x_{u1}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \dots, p\},\\
&x_{u2}^i \in [-5, 1],\ \ \forall\ i \in \{1, 2, \dots, r\},\\
&x_{l1}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \dots, q\},\\
&x_{l2}^i \in (0, e],\ \ \forall\ i \in \{1, 2, \dots, r\}.
\end{aligned} \tag{23}$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$\begin{aligned}
&x_{l1}^i = 0,\ \ \forall\ i \in \{1, 2, \dots, q\},\\
&x_{l2}^i = \log^{-1} x_{u2}^i,\ \ \forall\ i \in \{1, 2, \dots, r\}.
\end{aligned} \tag{24}$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

## 5.8 SMD8

This test problem tests the ability of the algorithms to handle multi-modality at the upper level, and convergence complexity at lower level at the same time. There is also a conflict between the upper level and lower level optimization tasks. The lower level objective contains the Rosenbrock's (banana) function, and the upper level objective contains the multi-modal Ackley's function. The constituent functions are chosen as

$$
\begin{aligned}
&F_1 = 20 + e - 20exp\left(-0.2\sqrt{\tfrac{1}{p}\sum_{i=1}^{p}(x_{u1}^i)^2}\right) - exp\left(\tfrac{1}{p}\sum_{i=1}^{p}\cos 2\pi x_{u1}^i\right),\\
&F_2 = -\sum_{i=1}^{q-1}\left(\left(x_{l1}^{i+1} - (x_{l1}^i)^2\right)^2 + (x_{l1}^i - 1)^2\right),\\
&F_3 = \sum_{i=1}^{r}(x_{u2}^i)^2 - \sum_{i=1}^{r}(x_{u2}^i - (x_{l2}^i)^3)^2,\\
&f_1 = \sum_{i=1}^{p}|x_{u1}^i|,\\
&f_2 = \sum_{i=1}^{q-1}\left(\left(x_{l1}^{i+1} - (x_{l1}^i)^2\right)^2 + (x_{l1}^i - 1)^2\right),\\
&f_3 = \sum_{i=1}^{r}(x_{u2}^i - (x_{l2}^i)^3)^2.
\end{aligned}
\tag{25}
$$

The range of variables is as follows:

$$
\begin{aligned}
&x_{u1}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \ldots, p\},\\
&x_{u2}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \ldots, r\},\\
&x_{l1}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \ldots, q\},\\
&x_{l2}^i \in [-5, 10],\ \ \forall\ i \in \{1, 2, \ldots, r\}.
\end{aligned}
\tag{26}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{aligned}
&x_{l1}^i = 1,\ \ \forall\ i \in \{1, 2, \ldots, q\},\\
&x_{l2}^i = (x_{u2}^i)^{\frac{1}{3}},\ \ \forall\ i \in \{1, 2, \ldots, r\}.
\end{aligned}
\tag{27}
$$

The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l$ is obtained by the relationship given above. Both upper and lower level functions are equal to zero at the optima.

## 5.9 SMD9

In this test problem, we introduce constraints at both upper and lower levels. Constraints are defined such that they cause convergence difficulties at both levels independently. One constraint is introduced at each level, such that the upper level constraint is a function of the upper level variables and the lower level constraint is a function of the lower level variables. The constraints divide the search space into annular regions, and cause convergence difficulties without altering the global optimum. The constraint at the upper as well as the lower level are however, inactive at the optimum. The two levels are once again conflicting in nature, such that an inaccurate lower level optimum may lead to upper level function value better than the true optimum for the bilevel problem. The constituent functions are chosen as

$$
\begin{aligned}
&F_1 = \sum_{i=1}^{p}(x_{u1}^i)^2,\\
&F_2 = -\sum_{i=1}^{q}(x_{l1}^i)^2,\\
&F_3 = \sum_{i=1}^{r}(x_{u2}^i)^2 - \sum_{i=1}^{r}(x_{u2}^i - \log(1 + x_{l2}^i))^2,\\
&f_1 = \sum_{i=1}^{p}(x_{u1}^i)^2,\\
&f_2 = \sum_{i=1}^{q}(x_{l1}^i)^2,\\
&f_3 = \sum_{i=1}^{r}(x_{u2}^i - \log(1 + x_{l2}^i))^2.
\end{aligned}
\tag{28}
$$

The upper and lower level constraints are as follows:

$$
\begin{array}{l}
\text{Upper level constraint}\\
G_1 : \frac{\sum_{i=1}^{p}(x_{u1}^i)^2+\sum_{i=1}^{r}(x_{u2}^i)^2}{a} - \left\lfloor \frac{\sum_{i=1}^{p}(x_{u1}^i)^2+\sum_{i=1}^{r}(x_{u2}^i)^2}{a} + \frac{0.5}{b} \right\rfloor \geq 0,\\
\text{Lower level constraint}\\
g_1 : \frac{\sum_{i=1}^{p}(x_{l1}^i)^2+\sum_{i=1}^{r}(x_{l2}^i)^2}{a} - \left\lfloor \frac{\sum_{i=1}^{p}(x_{l1}^i)^2+\sum_{i=1}^{r}(x_{l2}^i)^2}{a} + \frac{0.5}{b} \right\rfloor \geq 0,\\
\text{where } a = 1 \text{ and } b = 1.
\end{array}
\tag{29}
$$

The range of variables is as follows:

$$
\begin{array}{l}
x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\},\\
x_{u2}^i \in [-5, 1], \ \forall \ i \in \{1, 2, \ldots, r\},\\
x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\},\\
x_{l2}^i \in (-1, -1 + e], \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{30}
$$

Relationship between upper level variables (feasible with respect to upper level constraints) and lower level optimal variables is given as follows:

$$
\begin{array}{l}
x_{l1}^i = 0, \ \forall \ i \in \{1, 2, \ldots, q\},\\
x_{l2}^i = \log^{-1} x_{u2}^i - 1, \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{array}
\tag{31}
$$

Figure 10 shows the restricted search space for the upper level optimization task when it is a function of two upper level variables, i.e. $p = 1$ and $r = 1$. The search space looks similar at the lower level when $q = 1$ and $r = 1$. For higher number of variables, the annular rings transform into spherical shells. The values of the variables at the optima are $\mathbf{x}_u = 0$ and $\mathbf{x}_l = 0$. Both upper and lower level functions are equal to zero at the optima.

## 5.10 SMD10

In this test problem, we introduce constraints at the upper as well as the lower level such that they are scalable. As the number of variables are varied at the upper and the lower levels, the number of constraints also vary. This is different from the previous problem such that all the constraints are active at the optimum. However, in this case we have the upper level constraints as functions of the upper level variables, and the lower level constraints as functions of the lower level variables. The constituent functions are chosen as

$$
\begin{array}{l}
F_1 = \sum_{i=1}^{p}(x_{u1}^i - 2)^2,\\
F_2 = \sum_{i=1}^{q}(x_{l1}^i)^2,\\
F_3 = \sum_{i=1}^{r}(x_{u2}^i - 2)^2 - \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2,\\
f_1 = \sum_{i=1}^{p}(x_{u1}^i)^2,\\
f_2 = \sum_{i=1}^{q}(x_{l1}^i - 2)^2,\\
f_3 = \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2.
\end{array}
\tag{32}
$$

The upper and lower level constraints are as follows:

$$
\begin{array}{l}
\text{Upper level constraints}\\
G_j : x_{u1}^j - \sum_{i=1, i\neq j}^{p}(x_{u1}^i)^3 - \sum_{i=1}^{r}(x_{u2}^i)^3 \geq 0, \ \forall \, j \in \{1, 2, \ldots, p\},\\
G_{p+j} : x_{u2}^j - \sum_{i=1, i\neq j}^{r}(x_{u2}^i)^3 - \sum_{i=1}^{p}(x_{u1}^i)^3 \geq 0, \ \forall \, j \in \{1, 2, \ldots, r\},\\
\text{Lower level constraints}\\
g_j : x_{l1}^j - \sum_{i=1, i\neq j}^{q}(x_{l1}^i)^3 \geq 0, \ \forall \, j \in \{1, 2, \ldots, q\}.
\end{array}
\tag{33}
$$

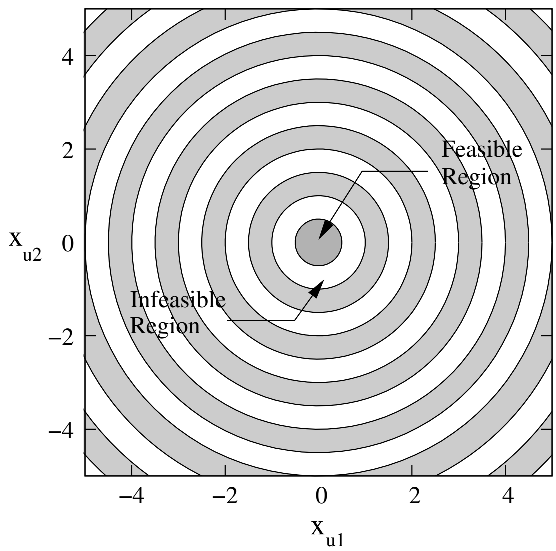



Figure 10: Feasible and infeasible regions in case of a two-variable constraint function.

The range of variables is as follows:

$$\begin{aligned} &x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\}, \\ &x_{u2}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, r\}, \\ &x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\}, \\ &x_{l2}^i \in (\tfrac{-\pi}{2}, \tfrac{\pi}{2}), \ \forall \ i \in \{1, 2, \ldots, r\}. \end{aligned} \tag{34}$$

Relationship between upper level variables (feasible with respect to upper level constraints) and lower level optimal variables is given as follows:

$$\begin{aligned} &x_{l1}^i = \tfrac{1}{\sqrt{q-1}}, \ \forall \ i \in \{1, 2, \ldots, q\}, \\ &x_{l2}^i = \tan^{-1} x_{u2}^i, \ \forall \ i \in \{1, 2, \ldots, r\}. \end{aligned} \tag{35}$$

The values of the variables at the optima are $\mathbf{x}_u = \frac{1}{\sqrt{p+r-1}}$, and $\mathbf{x}_l$ is obtained by the relationship given above.

Figure 11 shows the feasible region of the search space for the upper level optimization task, when the upper level objective it is a function of two upper variables, i.e. $p = 1, r = 1$. The shaded part in the figure shows the feasible region, and the dotted lines show the contours of the upper level objective function. For the given two variable upper level objective function, the optima lies at one of the intersections $((\mathbf{x}_{u1}, \mathbf{x}_{u2}) = (1, 1))$ of the constraints shown in the figure.

## 5.11 SMD11

In this test problem, we introduce constraints that are functions of upper as well as lower variables at both levels. The constraints at the upper level are scalable, but there is just a single constraint at the lower level. The constraint at the lower level introduces multiple global optimal solutions at the lower level for any given upper level vector. At the optimum of the bilevel problem, the lower level constraint

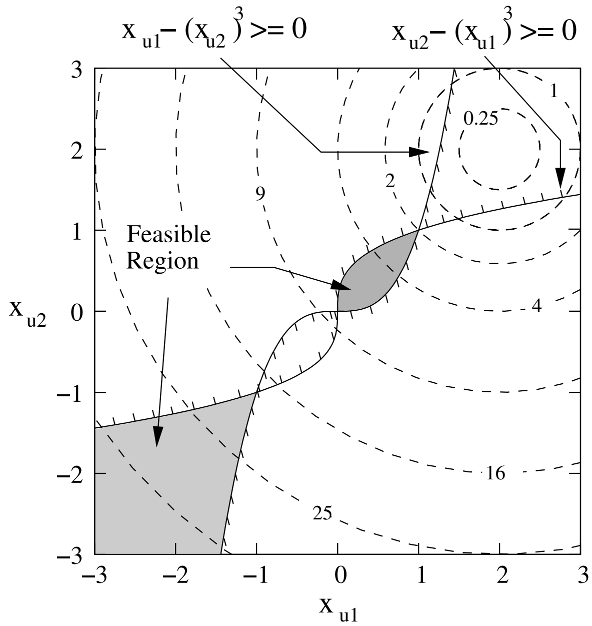



Figure 11: Feasible and infeasible regions in case of a two-variable constraint function.

as well as the upper level constraints are active. The upper level constraints eliminate a large part of the global optimal solutions from the lower level. The constituent functions are chosen as

$$
\begin{aligned}
F_1 &= \textstyle\sum_{i=1}^{p}(x_{u1}^i)^2, \\
F_2 &= -\textstyle\sum_{i=1}^{q}(x_{l1}^i)^2, \\
F_3 &= \textstyle\sum_{i=1}^{r}(x_{u2}^i)^2 - \sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2, \\
f_1 &= \textstyle\sum_{i=1}^{p}(x_{u1}^i)^2, \\
f_2 &= \textstyle\sum_{i=1}^{q}(x_{l1}^i)^2, \\
f_3 &= \textstyle\sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2.
\end{aligned}
\tag{36}
$$

The upper and lower level constraints are as follows:

$$
\begin{aligned}
&\text{Upper level constraints} \\
&G_j : x_{u2}^j \geq \tfrac{1}{\sqrt{r}} + \log x_{l2}^j, \ \forall\, j \in \{1, 2, \ldots, r\}, \\
&\text{Lower level constraint} \\
&g_1 : \textstyle\sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2 \geq 1.
\end{aligned}
\tag{37}
$$

The range of variables is as follows:

$$
\begin{aligned}
x_{u1}^i &\in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, p\}, \\
x_{u2}^i &\in [-1, 1], \ \ \forall \ i \in \{1, 2, \ldots, r\}, \\
x_{l1}^i &\in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
x_{l2}^i &\in [\tfrac{1}{e}, e], \ \ \forall \ i \in \{1, 2, \ldots, r\}.
\end{aligned}
\tag{38}
$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$
\begin{aligned}
x_{l1}^i &= 0, \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\
\mathbf{x}_{l2} &: \textstyle\sum_{i=1}^{r}(x_{u2}^i - \log x_{l2}^i)^2 = 1.
\end{aligned}
\tag{39}
$$

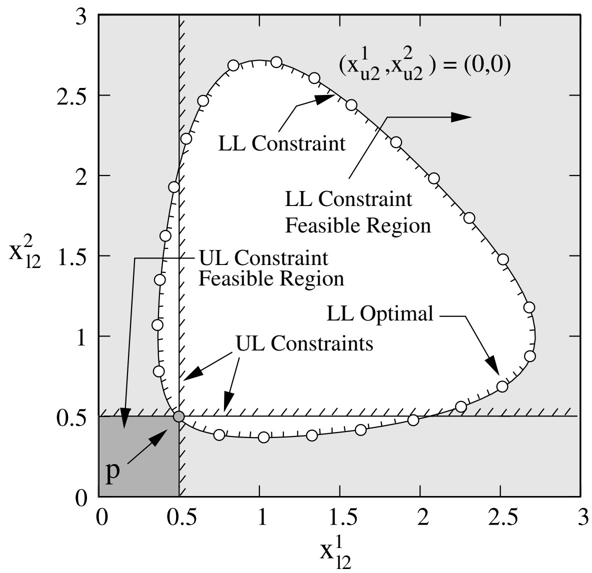



Figure 12: Feasible and infeasible regions of SMD11 for a particular upper level vector.

The values of the variables at the optima are $\mathbf{x}_{u1} = 0$, $\mathbf{x}_{u2} = 0$, $\mathbf{x}_{l1} = 0$, and $\mathbf{x}_{l2} = \log^{-1} \frac{-1}{\sqrt{r}}$. The upper level function value is $-1$ and the lower level function value is $+1$ at the optima.

Figure 12 shows the constraints at the upper as well as the lower level when $r = 2$. In this example, there is one constraint at the lower level and two constraints at the upper level. All the solutions on the lower level constraint represent optimal solutions to the lower level $f_3$. When $\mathbf{x}_{l1} = 0$, such that the function $f_2$ is also optimal, the solutions on the constraint are optimal solutions to the lower level problem for a given $\mathbf{x}_u$. It can be observed that the two constraints at the upper level eliminate all the lower level optimal solutions except one. The figure shows feasible region with respect to upper level constraints for the upper level problem. However, only point $\mathbf{p}$ represents a feasible solution for the upper level problem for a given $\mathbf{x}_u$, as it is the lower level optimal solution lying in the upper level constraint feasible region. This problem differs from SMD6, which also contained multiple global solutions at the lower level, in two ways. First, multiple global solutions at the lower level are introduced by lower level constraints in this problem, whereas in the previous problem it was the lower level objective function that was entirely responsible for introducing multiple global solutions. Second, out of the multiple global solutions from the lower level, a single solution is selected based on upper level constraints, whereas in the previous problem all the lower level global solutions were feasible but only one of those solutions had the best upper level objective value.

## 5.12 SMD12

This test problem is a combination of the previous two test problems, and involves a number of difficulties. The test problem has scalable constraints at both levels, and the constraints are functions of both upper as well as lower level variables. At the same time, any lower level optimization problem for a given set of upper level variables has multiple global optima. All the lower level constraints are active at the bilevel optimum. The constituent functions are chosen as

$$\begin{array}{l} F_1 = \sum_{i=1}^{p}(x_{u1}^i - 2)^2, \\ F_2 = \sum_{i=1}^{q}(x_{l1}^i)^2, \\ F_3 = \sum_{i=1}^{r}(x_{u2}^i - 2)^2 + \sum_{i=1}^{r}\tan|x_{l2}^i| - \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2, \\ f_1 = \sum_{i=1}^{p}(x_{u1}^i)^2, \\ f_2 = \sum_{i=1}^{q}(x_{l1}^i - 2)^2, \\ f_3 = \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2. \end{array} \tag{40}$$

The upper and lower level constraints are as follows:

$$\begin{array}{l} \text{Upper level constraints} \\ x_{u2}^i - \tan x_{l2}^i \geq 0, \ \forall\, i \in \{1, 2, \ldots, r\}, \\ x_{u1}^j - \sum_{i=1, i\neq j}^{p}(x_{u1}^i)^3 - \sum_{i=1}^{r}(x_{u2}^i)^3 \geq 0, \ \forall\, j \in \{1, 2, \ldots, p\}, \\ x_{u2}^j - \sum_{i=1, i\neq j}^{r}(x_{u2}^i)^3 - \sum_{i=1}^{p}(x_{u1}^i)^3 \geq 0, \ \forall\, j \in \{1, 2, \ldots, r\}, \\ \text{Lower level constraints} \\ \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2 \geq 1, \\ x_{l1}^j - \sum_{i=1, i\neq j}^{p}(x_{l1}^i)^3, \ \forall\, j \in \{1, 2, \ldots, q\}. \end{array} \tag{41}$$

The range of variables is as follows:

$$\begin{array}{l} x_{u1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, p\}, \\ x_{u2}^i \in [-14.10, 14.10], \ \ \forall \ i \in \{1, 2, \ldots, r\}, \\ x_{l1}^i \in [-5, 10], \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\ x_{l2}^i \in (-1.5, 1.5), \ \ \forall \ i \in \{1, 2, \ldots, r\}. \end{array} \tag{42}$$

Relationship between upper level variables and lower level optimal variables is given as follows:

$$\begin{array}{l} x_{l1}^i = \frac{1}{\sqrt{q-1}}, \ \ \forall \ i \in \{1, 2, \ldots, q\}, \\ \mathbf{x}_{l2} : \sum_{i=1}^{r}(x_{u2}^i - \tan x_{l2}^i)^2 = 1. \end{array} \tag{43}$$

The values of the variables at the optima are $\mathbf{x}_{u1} = \frac{1}{\sqrt{p+r-1}}$, $\mathbf{x}_{u2} = \frac{1}{\sqrt{p+r-1}}$, $\mathbf{x}_{l1} = \frac{1}{\sqrt{q-1}}$, and $\mathbf{x}_{l2} = \tan^{-1}(\frac{1}{\sqrt{p+r-1}} - \frac{1}{\sqrt{r}})$.

## 5.13 Summary and Precautions

The properties of the SMD test problems are summarized in Table 2. In the table, $N = \text{No}$ and $Y = \text{Yes}$. It can be observed that the 12 test problems are a good mix of various difficulties. We have tried to put the problems in an increasing order of difficulty. The last test problem can be observed to contain most of the difficulties except multi-modalities. This table should be helpful in testing algorithms for bilevel optimization. For example, if a new algorithm is able to solve SMD1 but not SMD2, one readily concludes that the algorithm is unable to handle a conflict. Similarly, if the algorithm is able to solve SMD1 and SMD2 but not SMD3 and SMD4, one would infer that the algorithm is unable to handle lower level multi-modality. Such information will be useful for an algorithm developer, as it helps him to identify the specific weaknesses in his approach, which he needs to improve on.

Authors would like to caution the developers against heavily relying on test problems alone to draw conclusions about the performance of the algorithm. The test problems are useful at the initial stages of algorithm development to evaluate the performance of an algorithm across various difficulty frontiers. However, it might not always be possible for a test-suite to provide difficulties that can be offered by complex real-world problems. Therefore, it is very important to note that the suggested test problems are not a replacement for realistic problems. It is important for researchers to focus on real-world problems as well along with the test suites to evaluate their procedures.

In the field of evolutionary multi-objective optimization, the test-suites have been quite famous and the developers are often found to draw strong conclusions based on the performance of the algorithms on these test-suites. One of the caveats is to exploit the structured nature of these test-suites to report better performance for their approaches. For example, in the proposed test-suite many of the test problems contain variable separable functions. These test problems would certainly be relatively easier to solve if an algorithm exploits this property of the test problems. Such algorithms would deteriorate drastically if these functions are rotated by multiplying the variables with a transformation matrix. On the other hand, an algorithm that does not exploit this property will be indifferent between the variable separable and the rotated test problems. It is important to utilize this knowledge about the test problems rather constructively to evaluate the extent to which an algorithm is exploiting the variable separability of the test problems. The authors would like the users to be careful about knowingly or unknowingly exploiting any such structure of the proposed test problems.

Table 2: Properties of SMD test problems.

| SMD | Upper Level | | | | Lower Level | | | | | Conflict |
|---|---|---|---|---|---|---|---|---|---|---|
| | Constrained | Scalability | | Multi-modality | Constrained | Scalability | | Multi-modality | Multiple Global Solutions | |
| | | Variables | Constraints | | | Variables | Constraints | | | |
| 1 | N | Y | - | N | N | Y | - | N | N | N |
| 2 | N | Y | - | N | N | Y | - | N | N | Y |
| 3 | N | Y | - | N | N | Y | - | Y | N | N |
| 4 | N | Y | - | N | N | Y | - | Y | N | Y |
| 5 | N | Y | - | N | N | Y | - | Y | N | Y |
| 6 | N | Y | - | N | N | Y | - | N | Y | Y |
| 7 | N | Y | - | Y | N | Y | - | N | N | Y |
| 8 | N | Y | - | Y | N | Y | - | Y | N | Y |
| 9 | Y | Y | N | N | Y | Y | N | N | N | Y |
| 10 | Y | Y | Y | N | Y | Y | Y | N | N | Y |
| 11 | Y | Y | Y | N | Y | Y | N | N | Y | Y |
| 12 | Y | Y | Y | N | Y | Y | Y | N | Y | Y |

# References


[1] A. Sinha, P. Malo, and K. Deb. Test problem construction for single-objective bilevel optimization. *Evolutionary Computation Journal*, 2013.